# MAC-CPTM Situations Project

## *Situation 15: Graphing Quadratic Functions*

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## Prompt

When preparing a lesson on graphing quadratic functions, a student teacher found that the textbook for the class claimed that $x = \frac{-b}{2a}$ was the equation for the line of symmetry of a parabola $y = ax^2 + bx + c$. The student teacher wondered how this equation was derived.

## Commentary

This prompt is centered on the graphing of quadratic equations, specifically the derivation of the equation of the line of symmetry. The foci in this situation deal with the general algebraic representation of any quadratic function, but they differ in the approaches used to obtain the equation in question. Focus 1 uses the symmetry of the parabola to find the $x$-coordinate of the vertex of the parabola. Focus 2 uses the first derivative to find the $x$-coordinate of the vertex of the parabola. Focus 3 utilizes transformations to the graph of $y = x^2$ to determine the coordinates of the vertex. Focus 4 uses some results about the roots of a polynomial equation, generally known as Viète's formulas to find the $x$-coordinate of the vertex of the parabola.

## Mathematical Foci

### *Mathematical Focus 1*

*The graph of a parabola is symmetric about its line of symmetry. Knowing the general form of the equation representing this graph can then allow us to identify the line of symmetry.*

A quadratic function can be written in the general form $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are real numbers and $a \neq 0$. The graph of this function is a parabola, symmetric about a line $x = k$, because its directrix is parallel to the $x$-axis. The symmetry of the graph about the line $x = k$ implies that:

$$a(k-1)^2 + b(k-1) + c = a(k+1)^2 + b(k+1) + c$$

Expanding the expressions on both sides of the equation yields:

$$ak^2 - 2ak + a + bk - b + c = ak^2 + 2ak + a + bk + b + c$$

Several like terms cancel, to simplify the equation to:

$$-2ak - b = 2ak + b$$

Solving for $k$ reveals that:

$$k = -\frac{b}{2a}$$

***Mathematical Focus 2***

*The first derivative of a polynomial function can be used to obtain the coordinates of the relative extrema of the function. In a parabola, this corresponds to the vertex whose $x$ coordinate gives the value for the line of symmetry.*

Polynomial functions are differentiable, and we can use the first derivative of the function to obtain critical values. These critical values will allow us to obtain the coordinates of the vertex of the parabola (the absolute minimum or maximum).

$$y = ax^2 + bx + c$$

$$y' = 2ax + b$$

To find the critical values, and thus the vertex and the equation of the line of symmetry, set the derivative equal to 0 and solve for $x$:

$$2ax + b = 0$$

$$x = \frac{-b}{2a}$$

This critical value is the $x$-coordinate of the vertex. Thus, the line of symmetry will be of the form $x = \frac{-b}{2a}$.

***Mathematical Focus 3***

*Using transformations, the graph of the function $y = x^2$ can be mapped to the graph of any quadratic function of the form $y = ax^2 + bx + c$.*

*The graph of the function given by $g(x) = v_1 f(h_2(x - h_1)) + v_2$ is the image of the graph of $f$ under the composition of a horizontal translation, a horizontal stretch, a vertical* *stretch, and a vertical translation related to the values of* $h_1$*,* $h_2$*,* $v_1$*, and* $v_2$*, respectively.*

The point $(0,0)$ is the vertex of the graph of the function given by $y = x^2$. To find the coordinates of the vertex of the general parabola, we apply the method known as completing the square to the general form of the equation of a parabola, which will yield a form given by the transformations noted above.

$$y = ax^2 + bx + c$$

$$y - c = ax^2 + bx$$

$$\frac{1}{a}(y-c) = x^2 + \frac{b}{a}x$$

$$\frac{1}{a}(y-c) + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2$$

$$\frac{1}{a}\left(y - c + \frac{b^2}{4a}\right) = \left(x + \frac{b}{2a}\right)^2$$

Now solving for *y*, we get a symbolic form of the equation that we can compare to $y = x^2$, the equation of the particular case, to get information about the needed transformation:

$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

The presence of $\left(x + \frac{b}{2a}\right)^2$ rather than $x^2$ implies a horizontal translation through $\frac{b}{2a}$ units in the negative direction (see Situation 30: Translation of Functions). This horizontal translation of the graph maps the vertex of the parabola from (0,0) to $\left(\frac{-b}{2a},0\right)$. The rest of the equation suggests a stretch (by a factor of *a)* and a vertical translation (through $\frac{b^2}{4a}$ units in the positive direction), neither of which affect the axis of symmetry. Thus the line of symmetry that passes through the vertex of the graph of the general parabola has the equation $x = \frac{-b}{2a}$.

***Mathematical Focus 4***

*General facts about the roots of polynomial equations, known as Viète's formulas, can quickly yield information about the line of symmetry of a parabola.*

Given the roots of a quadratic function, $r_1$ and $r_2$, a result of François Viète states that $r_1 + r_2 = -\frac{b}{a}$. However, the roots of a quadratic are symmetric about the vertex, thus $\frac{r_1 + r_2}{2} = -\frac{b}{2a}$ is the $x$ coordinate of the vertex and hence, $x = -\frac{b}{2a}$ is the equation of the line of symmetry of the parabola.